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16. Abstract

In calculating optimal transfers between Keplerian orbits, we are given a priori the maximum number of pulses, usually no more than three. Therefore, the question remains as to whether a greater number of pulses will reduce the characteristic velocity. The Pontryagin function is employed to describe a general case.

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ON THE PULSE NUMBER DURING OPTIMAL TRANSFER BETWEEN CLOSE KEPLERIAN ORBITS

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In calculating optimal transfers between Keplerian orbits, /165
we are previously given the maximum number of pulses, usually no more than three. Therefore, the question remains open as to whether an increase in the number of pulses can reduce the value of characteristic velocity. The solution of the problem on the most profitable method of transfer, when not only the pulse parameters but their number is optimized, can not be derived by a simple selection of all minimums. In such statement, the problem requires the application of methods of optimal synthesis associated with the principle of the maximum [1].

1. Statement of the Problem

Given x_i $(i = 1,\dots, 5)$--elements of a Keplerian orbit; and given at some point is applied an infinitely small pulse with a characteristic velocity equal to dv. Then increases in dx_i of the elements of the orbit will have the form

$$dx_i=f_i(\bar{x}, \alpha_1, \alpha_2, \alpha_3, \varphi)\,dv,$$

where $\bar{x} = (x_1,\dots, x_5)$; a_1, a_2, a_3--direction cosines of the pulse; ϕ--true anomaly of the point of application of the pulse.

Switching to a differentiation with respect to the characteristic velocity v, we can formula the problem of pulse transfer between orbits in the form of a problem of speed of response

$$\frac{dx_i}{dv}=f_i(\bar{x}, \alpha_1, \alpha_2, \alpha_3, \varphi) \qquad (i=1, \dots, 5). \tag{1.1}$$

The direction parameters are direction cosines and true anomaly; the phase coordinates are orbital elements; the argument--characteristic velocity. The range of control is defined by the relationships

$$\alpha_1^2+\alpha_2^2+\alpha_3^2=1, \quad -\pi<\varphi\leqslant\pi.$$

Let us solve the problem assuming that

$$f_i(\bar{x}, \alpha_1, \alpha_2, \alpha_3, \varphi)=f_i(\bar{x}_0, \alpha_1, \alpha_2, \alpha_3, \varphi). \tag{1.2}$$

where $\bar{x}_0$ are elements of the initial orbit, i.e., that f_i is not dependent on the phase coordinates. This approach corresponds to a transfer between infinitely close orbits.

2. General Case /166

Following the principle of the maximum [1], let us introduce auxiliary variables λ_i and write the Pontryagin function

$$\Pi=\sum_{i=1}^{5} \lambda_i f_i, \tag{2.1}$$

wherein $\sum_{i=1}^{5} \lambda_i^2 \neq 0$. Due to condition (1.2), the values of ${}_i$ will be constant. The number of pulses which are used for the transfer is equal to the number q of different combinations of a_1, a_2, a_3, ϕ, providing functions Π with given λ_i and $\bar{x}_0$ with an absolute minimum.

Let us aim at defining the function

$$p(\bar{x}_0)=\max_{\bar{\lambda}} q(\bar{\lambda}, \bar{x}_0). \text{ где } \bar{\lambda}=(\lambda_1, \ldots, \lambda_5).$$

Let us define the elements x_i as follows:

$$
\begin{aligned}
&x_1=\mu^{1/2}a^{-1/2}, \quad x_2=\mu^{1/2}p^{-1/2}, \quad x_3=\mu^{1/2}a^{-1/2}e\omega,\\
&x_4=\mu^{-1/2}p_0^{1/2}\Omega, \quad x_5=\mu^{-1/2}p_0^{1/2}\gamma.
\end{aligned} \tag{2.2}
$$

where μ--field constant; q--semimajor axis of the orbit; p--focal parameter; p_0--focal parameter of the initial orbit; e--eccentricity; Ω--length of ascending node; ω--latitude of the pericenter; γ --orbital inclination.

The system of coordinates will be selected so that the inclincation of the initial orbit is equal to $\pi/2$, and the latitude of the pericenter is zero. Then system (1.1), due to condition (1.2), will have the form

$$
\begin{aligned}
&\frac{dx_1}{dv}=-e(1-e^2)^{-1/2}\sin\varphi\alpha_1-(1-e^2)^{-1/2}(1+e\cos\varphi)\alpha_2,\\
&\frac{dx_2}{dv}=-(1+e\cos\varphi)^{-1}\alpha_2,\\
&\frac{dx_3}{dv}=-(1-e^2)^{1/2}\cos\varphi\alpha_1+(1-e^2)^{1/2}(1+e\cos\varphi)^{-1}(2+e\cos\varphi)\sin\varphi\alpha_2,\\
&\frac{dx_4}{dv}=(1+e\cos\varphi)^{-1}\sin\varphi\alpha_3,\\
&\frac{dx_5}{dv}=(1+e\cos\varphi)^{-1}\cos\varphi\alpha_3.
\end{aligned} \tag{2.3}
$$

Let us note that the right sides of (2.3) depend only on the control parameters and eccentricity of the initial orbit; consequently, $p(\overline{x}_0) = p(e_0)$.

Let us now write the Pontryagin function

$$
\begin{aligned}
\Pi=&-\lambda_1(1-e^2)^{-1/2}(e\sin\varphi\alpha_1+(1+e\cos\varphi)\alpha_2)-\lambda_2(1+e\cos\varphi)^{-1}\alpha_2-\\
&-\lambda_3(1-e^2)^{1/2}(\cos\varphi\alpha_1-\sin\varphi(1+e\cos\varphi)^{-1}(2+e\cos\varphi)\alpha_2)+\\
&+\lambda_4(1+e\cos\varphi)^{-1}\sin\varphi\alpha_3+\lambda_5(1+e\cos\varphi)^{-1}\cos\varphi\alpha_3.
\end{aligned} \tag{2.4}
$$

since $a_1^2 + a_2^2 + a_3^2 = 1$, then

$$M(\bar{\lambda}, \varphi, e) = (\max_{\alpha_1, \alpha_2, \alpha_3} \Pi)^2 = $$
$$= \lambda_1^2 (1-e^2)^{-1} (1+2e\cos\varphi + e^2) + 2\lambda_1\lambda_2 (1-e^2)^{1/2} +$$

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$$+\lambda_2^2 (1+e\cos\varphi)^{-2} - 4\lambda_1\lambda_3 \sin\varphi - 2\lambda_2\lambda_3 (1-e^2)^{1/2} (1+e\cos\varphi)^{-2} (2+e\cos\varphi)\sin\varphi +$$
$$+\lambda_3^2 (1-e^2)(1+e\cos\varphi)^{-2} (4+1/e\cos\varphi + (e^2-3)\cos^2\varphi - 2e\cos^3\varphi) +$$
$$+(\lambda_4 \sin\varphi + \lambda_5 \cos\varphi)^2 (1+e\cos\varphi)^{-2}. \quad (2.5)$$

The quantities a_1, a_2, a_3 are defined uniquely, as soon as ϕ is defined. Thus, the problem was reduced to an explanation of how many different values of $\phi (-\pi < \phi \leq \pi)$ can provide functions M at given e and $\bar{\lambda}$ with an absolute maximum.

Let us define c = max M. Let us consider the equation with respect to ϕ:

$$M - c = 0. \quad (2.6)$$

Given $-\pi<\phi<\pi$. With the aid of the statement

$$\cos\varphi = \frac{1-y^2}{1+y^2}, \quad \sin\varphi = \frac{2y}{1+y^2} \quad (2.7)$$

equation (2.5) is reduced to an equation of sixth power with respect to y. Each real root of this equation is at least double (by definition c). Consequently, the number of different real roots is no more than three.

Now let functions of M be provided with a maximum of $\phi = \pi$. In this case, permutation of (2.7) leads to equation of the fourth power and consequently, again $p(e) \leq 3$.

Let us show that p(e) = 3 if 0 < e < 1. We will assume that

$$\left.\begin{aligned}\lambda_1&=e^{-1/2}(1+e)^{-1/2}(1-e)^{1/2},\\ \lambda_2&=e^{-1/2}(1+\delta)^{1/2}(1-2e-\delta)^{1/2},\\ \lambda_3&=\lambda_5=0,\\ \lambda_4&=e^{1/2}(1+e)^{-1}(1+\delta)^{1/2}(3+2e+\delta)^{1/2},\end{aligned}\right\} \tag{2.8}$$

where

$$-e<\delta<-2e+1. \tag{2.9}$$

We can verify that due to (2.9), the radicands appearing in (2.8) are positive; moreover, the values of $\phi = 0$, ϕ = arccos (δ/e), ϕ= -arccos(δ/e) give function M an absolute maximum.

3. Coplanar Transfer

Assume that

$$\lambda_4=\lambda_5=0, \tag{3.1}$$

then a_3 = 0, which corresponds to a coplanar transfer. Let us designate that

$$m(e)=\max q(\lambda_1,\lambda_2,\lambda_3,0,0,e). \tag{3.2}$$

Apparently, m(e) $\leq$ p(e). For precise definition of m(e), let us again apply in the interval $-\pi<\phi<\pi$ the permutation of (2.7) which with allowance for (3.1) brings equation (2.5) again to an equation of the sixth power. If this equation has three double roots, the corresponding polynomial is a complete square of the polynomial of the third power. This condition permits us to define λ_1, λ_2, λ_3 with an accuracy to within the constant coefficient, and the polynomial of the third power has the form

$$(5+t)\,y^3+2\,(1+t)^{1/2}\,y^2+(2+5t-t^2)\,y+(8+4t)\,(1+t)^{1/2}=0, \tag{3.3}$$

where $t = (1 + e)/(1 - e)$, or differs from (3.3) only by replacement of y by -y. The discriminant of equation (3.3) within unessential coefficients is equal to

$$t^4-22t^3-75t^2-472t-404 \tag{3.4}$$

and changes sign when $t^* = 25.665\ldots$ When $t < t^*$, as is easily seen in the case of $t = 1$, equation (3.3) has a pair of complex roots. Thus, $m(e) = 3$ only for $e > e^*$, where $e^* = (t^* - 1)/(t^* + 1) = 0.925\ldots.$ For completeness, let us note that if $\phi = \pi$ was one of the values providing the maximum M, the problem by analogy would reduce to representation of a polynomial of the fourth power in the form of the square of a polynomial of the second power, but no corresponding set of essential λ_1, λ_2, λ_3 exists. /168

When $e < e^*$, $m(e) = 2$, since for non-intersecting orbits the optimal transfer is at least two-pulse, and it can not be three pulse as proven.

In the work of Krasinsky the fact is noted that pulses making up a three-pulse transfer must be applied in strictly fixed points of the orbit, but the region itself of threepulse transfers could not be found. This is understandable since the Krasinsky method is based on the decomposition by powers of e, and e^*, as we have seen, lies even beyond the Laplace limit.

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